Homogenization II

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Outline

- In many classical problems: diffusion, electrical conduction, linear elasticity, homogenized equations have the same structure as fine scale problems. Example: linear elasticity upscales to a linear elasticity system with effective elastic modulae.
- Today, we'll look at two problems of a different sort. The structure of the effective equations for these problems is (quite) different from the structure of the ε-problems.

Abstractions

- Reality: a finite size sample of one specific material. Scales are separated: $l \ll L$. Introduce a small parameter $\varepsilon = \frac{l}{L}$. Our material corresponds to one specific value of ε.
- Abstraction: consider infinitely many fictitious materials, corresponding to infinitesimally small values of ε. Analogy: transition from small increments to infinitesimally small increments in calculus.
- Another way to think of these materials is in the spirit of statistical physics. Infinite set of materials is an ensemble. Question: how to generate such an ensemble? Easy if our material has a periodic microstructure. Then l is the size of the periodicity cell $\frac{l}{L}Y$. Once we have the cell, we can shrink it an fill the original domain with the shrunk copies of εY of a fixed (unit) cell Y.
- Next step: approximate the physical fields (strain, stress) in an actual material with the $\varepsilon \to 0$ limits of the corresponding ensemble quantities.

More abstractions

- Two–scale asymptotic expansions.
  In periodic homogenization problems, it often makes sense to approximate functions like this:
  $$f(x,\varepsilon) \approx f^0\left(x,\frac{x}{\varepsilon}\right) + \varepsilon f^1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon^2 f^2\left(x,\frac{x}{\varepsilon}\right) + \ldots,$$
  where $f^j(x,y), j = 1,2,\ldots$ are periodic with respect to y.
- Enlarge dimension of the problem:
  $$f(x,\varepsilon) \to f(x,y)_{|y=\frac{x}{\varepsilon}}.$$
  This suggest studying functions $f(x,y)$ on $\Omega \times Y$, where Ω is the "physical" domain, and Y is the periodicity cell.
- This "lifting" leads to separation of slow (x) and fast (y) variables. Also, the x,y-problems are parameter independent. The goal: separate variables, solve a small number of fast variable cell problems posed on Y, and then use the results to define effective quantities in the slow variable effective equations.

Flow in porous media

Notation:

–Ω-a physical domain modeling a porous composite material;

– Ω^ε- a family of periodic perforated domains modeling pore space in a rigid porous matrix; $\partial\Omega^\varepsilon$ is the boundary of Ω^ε;

–Y-periodicity cell; Y_f-pore space part of the periodicity cell;

– $\boldsymbol{v}^\varepsilon$-fluid velocity;

– P^ε -fluid pressure;

– $\boldsymbol{f}$-density of a body force (e.g. gravity);

– μ -viscosity of the fluid;

The flow is governed by Stokes equations:

$$\mu\Delta\boldsymbol{v}^\varepsilon - \nabla P^\varepsilon = \boldsymbol{f}, \quad \text{in } \Omega^\varepsilon, \tag{1}$$

$$\text{div } \boldsymbol{v}^\varepsilon = 0, \quad \text{in } \Omega^\varepsilon, \tag{2}$$

$$\boldsymbol{v}^\varepsilon = 0, \quad \text{on } \partial\Omega^\varepsilon. \tag{3}$$

Two-scale asymptotic expansions

$$\boldsymbol{v}^\varepsilon = \varepsilon^2 \boldsymbol{v}^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 \boldsymbol{v}^1\left(x, \frac{x}{\varepsilon}\right) + \ldots, \tag{4}$$

$$P^\varepsilon = P^0(x) + \varepsilon P^1\left(x, \frac{x}{\varepsilon}\right) + \ldots. \tag{5}$$

Begin working with (2). Use

$$\operatorname{div} f\left(x, \frac{x}{\varepsilon}\right) = \left(\operatorname{div}_x + \frac{1}{\varepsilon}\operatorname{div}_y\right) f(x,y)|_{y=\frac{x}{\varepsilon}}, \tag{6}$$

Plug this into (4) and collect like terms:

$$0 = \varepsilon \operatorname{div}_y \boldsymbol{v}^0 + \varepsilon^2\left(\operatorname{div}_x \boldsymbol{v}^0 + \operatorname{div}_y \boldsymbol{v}^1\right) + \ldots.$$

Therefore

$$\operatorname{div}_y \boldsymbol{v}^0 = 0, \tag{7}$$

$$\operatorname{div}_x \boldsymbol{v}^0 + \operatorname{div}_y \boldsymbol{v}^1 = 0, \tag{8}$$

Averaging operator

$$\overline{\boldsymbol{v}} \equiv \frac{1}{|Y|}\int_Y \boldsymbol{v}(x,y)dy.$$

Effective velocity is defined as $\overline{\boldsymbol{v}}^0$.

Apply the averaging operator to (8).

$$\begin{aligned} 0 &= \mathrm{div}_x \overline{\boldsymbol{v}}^0 + \frac{1}{|Y|}\int_Y \mathrm{div}_y \boldsymbol{v}^1(x,y)dy \\ &= \mathrm{div}_x \overline{\boldsymbol{v}}^0 + \frac{1}{|Y|}\int_{Y_f} \mathrm{div}_y \boldsymbol{v}^1(x,y)dy \\ &= \mathrm{div}_x \overline{\boldsymbol{v}}^0 + \frac{1}{|Y|}\int_{\partial Y_f} \boldsymbol{v}^1(x,y)\cdot\boldsymbol{\nu}(y)dy \\ &= \mathrm{div}_x \overline{\boldsymbol{v}}^0. \end{aligned} \tag{9}$$

The first effective equation:

$$\mathrm{div}_x \overline{\boldsymbol{v}}^0 = 0.$$

Effective flow is incompressible.

Next, use (4), (5) in (2).
ε^0 -term:

$$\mu \Delta_y \boldsymbol{v}^0 - \nabla_y P^1 = \boldsymbol{f} - \nabla_x P^0. \tag{10}$$

Note that RHS depends only on x, and the differentiations in LHS are in y.

Idea: separate variables and look for $\boldsymbol{v}^0$ of the form

$$\boldsymbol{v}^0(x,y) = \frac{1}{\mu}\left(f_i - \partial_{x_i} P^0\right)(x)\boldsymbol{w}^i(y). \tag{11}$$

Summation with respect to $i = 1, 2, 3$, and $\boldsymbol{w}^i$ are vectors to be determined.

Insert (11) into (10). Componentwise,

$$\left(f_i - \partial_{x_i} P^0\right) \Delta w_j^i - \partial_{y_j} P^1 = f_j - \partial_{x_j} P^0 = \left(f_i - \partial_{x_i} P^0\right) \delta_{ij}, \quad j = 1, 2, 3.$$

Summation in i, and j is fixed. Also write

$$P^1(x, y) = \left(f_i - \partial_{x_i} P^0\right)(x) P^{1,i}(y)$$

Rearrange:

$$\left(f_i - \partial_{x_i} P^0\right)\left(\Delta w_j^i - \delta_{ij} - \partial_{y_j} P^{1,i}\right) = 0, \quad j = 1, 2, 3.$$

This will be solved if $\boldsymbol{w}^i(y)$ satisfy

Cell problems

$$\begin{aligned} &\Delta \boldsymbol{w}^i - \nabla P^{1,i} = \mathbf{e}_i, \ \text{ in } Y_f, \\ &\operatorname{div} \boldsymbol{w}^i = 0 \ \text{ in } Y_f \end{aligned} \tag{12}$$

Boundary conditions: periodic on $\partial Y_f \cap \partial Y$, zero on the rest of ∂Y_f.

Final step: apply averaging operator to (11). This yields

Darcy law

$$\overline{\boldsymbol{v}}^0 = \boldsymbol{K}(\boldsymbol{f} - \nabla P^0), \tag{13}$$

where the effective permeability tensor $\boldsymbol{K}$ is defined by

$$K_{ij} = \frac{1}{\mu}\overline{w}^i_j. \tag{14}$$

Complete set of effective equations

$$\overline{\boldsymbol{v}}^0 = \boldsymbol{K}(\boldsymbol{f} - \nabla P^0),$$
$$\operatorname{div} \overline{\boldsymbol{v}}^0 = 0,$$

where $\boldsymbol{K}$ is defined by (14), and w_j^i are components of vectors $\boldsymbol{w}^i$ satisfying cell problems (12). Elimination of $\boldsymbol{v}^0$ yields a diffusion equation for P^0

$$-\operatorname{div}\left(\boldsymbol{K}\nabla P^0\right) = F, \tag{15}$$

where $F = -\operatorname{div}(\boldsymbol{K}\boldsymbol{f})$. Boundary conditions for (15)

$$\left(\boldsymbol{K}\nabla P^0\right) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega.$$

Acoustics of solid-fluid mixtures

The composite Ω is a union of a solid Ω_s^ε and fluid Ω_f^ε domains.

Solid equations.

$$\rho^s \boldsymbol{v}_t - \operatorname{div} \boldsymbol{T}^s = \boldsymbol{f}, \qquad \text{in } \Omega_s^\varepsilon. \tag{16}$$

$$\boldsymbol{T}^s = \boldsymbol{A} e(\boldsymbol{u}) + \boldsymbol{B}^s e(\boldsymbol{v}). \tag{17}$$

Here $\boldsymbol{u}$ is displacement, $\boldsymbol{v}$ velocity, $\boldsymbol{A}$ is the material stiffness tensor, $\boldsymbol{B}^s$ is the viscosity tensor, $e = \frac{1}{2}(\nabla + \nabla^T)$, so that $e(\boldsymbol{u})$ is strain.

Fluid equations.

$$\rho^f \boldsymbol{v}_t - \mathrm{div}\ \boldsymbol{T}^f = \boldsymbol{f}, \qquad \text{in } \Omega_f^\varepsilon. \tag{18}$$

$$\boldsymbol{T}^f = \boldsymbol{B}^f e(\boldsymbol{v}) + c^2 \rho^f \mathrm{div} \boldsymbol{u}\ \boldsymbol{I}. \tag{19}$$

These are the constitutive equations of a slightly compressible viscous fluid. Here $\boldsymbol{B}^f$ is the viscosity tensor, c is the speed of sound. Comment on the derivation of (19).

In the periodicity cell Y, define the solid Y^s and fluid Y^f parts. Next, let

$$\rho(y) = \begin{cases} \rho^s & \text{in } Y^s, \\ \rho^f & \text{in } Y^f, \end{cases}$$

Composite stiffness

$$\boldsymbol{A}(y) = \begin{cases} \boldsymbol{A}^s & \text{in } Y^s, \\ c^2 \rho^f \boldsymbol{I} & \text{in } Y^f, \end{cases}$$

Composite viscosity

$$\boldsymbol{B}(y) = \begin{cases} \boldsymbol{B}^s & \text{in } Y^s, \\ \boldsymbol{B}^f & \text{in } Y^f, \end{cases}$$

Define the composite density, stiffness and viscosity $\rho^\varepsilon = \rho\left(\frac{x}{\varepsilon}\right)$, $\boldsymbol{A}^\varepsilon = \boldsymbol{A}\left(\frac{x}{\varepsilon}\right)$, $\boldsymbol{B}^\varepsilon = \boldsymbol{B}\left(\frac{x}{\varepsilon}\right)$. Combine (16) and (18) into one system

$$\rho^\varepsilon \boldsymbol{u}^\varepsilon_{tt} - \operatorname{div}\left(\boldsymbol{A}^\varepsilon e(\boldsymbol{u}^\varepsilon) + \boldsymbol{B}^\varepsilon e(\boldsymbol{u}^\varepsilon_t)\right) = \boldsymbol{f}, \qquad \text{in } \Omega. \tag{20}$$

Two-scale expansion

$$\boldsymbol{u}^\varepsilon(t,x) = \boldsymbol{u}^0(t,x) + \varepsilon \boldsymbol{u}^1\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon^2 \boldsymbol{u}^2\left(t,x,\frac{x}{\varepsilon}\right) + \dots. \tag{21}$$

To lift, notice that for each smooth enough $\boldsymbol{w}\left(t, x, \frac{x}{\varepsilon}\right)$,

$$e\left(\boldsymbol{w}\left(t, x, \frac{x}{\varepsilon}\right)\right) = [e_x + \frac{1}{\varepsilon} e_y](\boldsymbol{w}(x, y))_{|y=\frac{x}{\varepsilon}}, \tag{22}$$
$$\operatorname{div}\left(\boldsymbol{w}\left(t, x, \frac{x}{\varepsilon}\right)\right) = [\operatorname{div}_x + \frac{1}{\varepsilon}\operatorname{div}_y]\boldsymbol{w}(x, y)_{|y=\frac{x}{\varepsilon}}.$$

Plug (21) into (20), take (22) into account and collect like terms.

ε^{-1}–term and ε^{0}–term, are, respectively

$$\mathrm{div}_y \boldsymbol{T}^0 = 0, \tag{23}$$

$$\rho(y)\boldsymbol{u}^0_{tt} - \mathrm{div}_x \boldsymbol{T}^0 - \mathrm{div}_y \boldsymbol{T}^1 = \boldsymbol{f}, \tag{24}$$

where

$$\boldsymbol{T}^0 = \boldsymbol{A}(y)\left(e(\boldsymbol{u}^0) + e_y(\boldsymbol{u}^1)\right) + \boldsymbol{B}(y)\left(e(\boldsymbol{u}^0_t) + e_y(\boldsymbol{u}^1_t)\right),$$

$$\boldsymbol{T}^1 = \boldsymbol{A}(y)\left(e_x(\boldsymbol{u}^1) + e_y(\boldsymbol{u}^2)\right) + \boldsymbol{B}(y)\left(e_x(\boldsymbol{u}^1_t) + e_y(\boldsymbol{u}^2_t)\right).$$

ε^{-1}–term: ansatz

A good ansatz:

$$\begin{aligned} \boldsymbol{u}^1(t,x,y) &= \boldsymbol{n}^{pq}(y)e(\boldsymbol{u}^0)_{pq}(t,x) + \\ &\int_0^t \boldsymbol{m}^{pq}(t-\tau,y)e(\boldsymbol{u}_t^0)_{pq}(\tau,x)d\tau. \end{aligned} \tag{25}$$

Then

$$\begin{aligned} \boldsymbol{A}e_y(\boldsymbol{u}^1) + \boldsymbol{B}e_y(\boldsymbol{u}_t^1) &= \mathcal{F}_{11}^{pq}e(\boldsymbol{u}^0)_{pq} + \mathcal{F}_{12}^{pq}e(\boldsymbol{u}_t^0)_{pq} \\ &+\mathcal{F}_2^{pq}e(\boldsymbol{u}_t^0)_{pq} + \int_0^t \mathcal{F}_3^{pq}(t-\tau)e(\boldsymbol{u}_t^0)_{pq}(\tau)d\tau, \end{aligned} \tag{26}$$

In (26),

$$\mathcal{F}_{11}^{pq} = \boldsymbol{A}\, \boldsymbol{I}^{pq} + \boldsymbol{A}^s \theta(y) e_y(\boldsymbol{n}^{pq}), \tag{27}$$

$$\mathcal{F}_{12}^{pq} = \boldsymbol{B}^f (1 - \theta(y)) e_y(\boldsymbol{n}^{pq}), \tag{28}$$

$$\mathcal{F}_{2}^{pq} = c^2 \rho^f \boldsymbol{I}\, e(\boldsymbol{n}^{pq}) + \boldsymbol{B}(\boldsymbol{I}^{pq} + e_y(\boldsymbol{m}_0^{pq})), \tag{29}$$

$$\mathcal{F}_{3}^{pq} = \boldsymbol{A} e(\boldsymbol{m}^{pq}) + \boldsymbol{B} e(\boldsymbol{m}_t^{pq}). \tag{30}$$

$$\boldsymbol{I}^{pq} = \frac{1}{2}(\mathbf{e}_p \otimes \mathbf{e}_q + \mathbf{e}_q \otimes \mathbf{e}_p),$$

and

$$\theta(y) = \begin{cases} 1, & \text{if } y \in Y^s, \\ 0, & \text{if } y \in Y^f. \end{cases}$$

Cell problems

$$\mathrm{div}_y \mathcal{F}^{11} = \mathrm{div}_y \mathcal{F}^{12} = 0. \tag{31}$$

This is an elasticity-type problem for $\boldsymbol{n}^{pq}$.

$$\mathrm{div}_y \mathcal{F}^{2} = 0. \tag{32}$$

This is an elasticity-type problem for $\boldsymbol{m}_0^{pq}$.

$$\mathrm{div}_y \mathcal{F}^{3} \equiv \mathrm{div}_y \left[\boldsymbol{A} e(\boldsymbol{m}^{pq}) + \boldsymbol{B} e(\boldsymbol{m}_t^{pq}) \right] = 0. \tag{33}$$

This is an evolution equation for $\boldsymbol{m}^{pq}(t, y)$ with the initial condition $\boldsymbol{m}^{pq}(0, y) = \boldsymbol{m}_0^{pq}$.
The sequence: first solve (31). Then use this solution in the source term in (32), and find $\boldsymbol{m}_0^{pq}$. Then use this solution as the initial condition in (33) and find $\boldsymbol{m}^{pq}$.

Effective equation

Apply the averaging operator to (24) and observe that

$$\overline{\rho(y)\boldsymbol{u}^0_{tt}(t,x)} = \overline{\rho}\boldsymbol{u}^0_{tt},$$

and

$$\overline{\mathrm{div}_y \boldsymbol{T}^1} = \frac{1}{|Y|}\int_Y \mathrm{div}_y \boldsymbol{T}^1 dy = \frac{1}{|Y|}\int_{\partial Y} \boldsymbol{T}^1 \boldsymbol{\nu} dy = 0$$

by Y-periodicity.

Effective equation:

$$\overline{\rho}\boldsymbol{u}^0_{tt} - \mathrm{div}\,\overline{\boldsymbol{T}} = \boldsymbol{f},$$

where the effective constitutive equation is

$$\overline{\boldsymbol{T}} = \overline{\boldsymbol{A}}e(\boldsymbol{u}^0) + \overline{\boldsymbol{B}}e(\boldsymbol{u}^0_t) + \int_0^t \overline{\boldsymbol{C}}(t-\tau)e(\boldsymbol{u}^0_t)(\tau)d\tau.$$

Effective material properties

$$\overline{A}_{ijpq} = \overline{\mathcal{F}}^{pq}_{11,ij},$$
$$\overline{B}_{ijpq} = \overline{\mathcal{F}}^{pq}_{12,ij} + \overline{\mathcal{F}}^{pq}_{2,ij},$$
$$\overline{C}_{ijpq}(t) = \overline{\mathcal{F}}^{pq}_{3,ij}(t).$$

Conclusions

- two-scale expansions work, at least on the formal level, in a variety of linear homogenization problems;
- two-scale techniques deliver explicit formulas for the effective properties;
- the details of the postulated expansions (e.g scaling of the first term, dependence of the first term only on x) are problem-specific;
- the crucial part of the method is a problem-specific ansatz which expresses $\boldsymbol{u}^1$ in terms of $\boldsymbol{u}^0$. Compare with the closure problem in other asymptotic methods (e.g. statistical mechanics);
- the effective material response of solid-fluid mixtures is often different from the material response of both constituents.

Further reading

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3. A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic Analysis For Periodic Structures*. Elsevier North-Holland, 1978.